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THE THREE-DIMENSIONAL CONTACT PROBLEM FOR AN ELASTIC WEDGE TAKING FRICTION FORCES INTO ACCOUNT†

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The method employed in [1] is used to solve the first fundamental three-dimensional problem of the theory of elasticity for a wedge. This consists of reducing it, using a complex Fourier–Kontorovich–Lebedev integral, to a generalized Hilbert boundary-value problem, as generalized by Vekua. Formulae are given which enable one to calculate the displacement vector and the stress tensor completely in a three-dimensional elastic wedge, one face of which is stress-free, while a normal and shear load (perpendicular to the edge) act on the other. Using the solution obtained, the contact problem of the motion of a punch on the face of an elastic wedge in a direction perpendicular to the wedge edge is considered (in the quasi-static formulation). The punch is considerably elongated along the edge of an elliptic paraboloid, and hence it can be assumed approximately that the friction forces are collinear with the direction of motion. The effect of the Coulomb friction coefficient on the relation between the impressing force and the settlement of the punch for different wedge angles is investigated. The effective stress on the axis of symmetry of the contact region is calculated for different wedge angles and as a function of the distance of the punch from the wedge edge and also as a function of the direction and value of the friction forces.

The method used below has been employed [2] to solve the problem of the action of a normal load on one face of a three-dimensional wedge for various boundary conditions on the other face. This method has also been used [3] to obtain a solution of the problem of the action of a normal and shear load (perpendicular to the edge of the wedge) on one face of a wedge, and an expression has been derived for the normal displacement on this face. The formulae obtained below generalize the well-known solutions of the Boussinesq and Cerruti problems for a half-space [4], while the formulation of the contact problem generalizes the well-known case of the motion of a punch on a half-space [5]. To solve the integral equation of the contact problem with unknown contact region the method of non-linear boundary integral equations [6,7] is used, which enables one, simultaneously and fairly rapidly, to determine the required contact pressures and the contact area. Variational methods, which are more universal, were employed to solve three-dimensional contact problems with friction in [8, 9]. The results obtained for an elastic wedge are important in applications to Novikov gears [10,11].

1. THE THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE

Consider a three-dimensional elastic wedge with aperture angle α and elastic characteristics ν (Poisson's ratio) and G (the shear modulus) in cylindrical coordinates r, φ, z, where the z axis is directed along the edge of the wedge so that the system of coordinates is a right-handed system. Suppose the face of the wedge $\varphi = -\alpha/2$ is stress-free, while a normal load of intensity $q(r, q)$, distributed over a finite region Ω, acts on the face $\varphi = \alpha/2$, while a shear load, perpendicular to the edge of the wedge, distributed over the same region and proportional to the normal load with a coefficient of proportionality μ, also acts on this face. We will assume, for simplicity, that the region Ω is symmetrical about the semiaxis $z = 0$ and $q(r, -z) = q(r, -z)$. We will write the boundary conditions in the form

$$\begin{aligned} &\varphi=-\alpha/2: \quad \sigma_\varphi=\tau_{r\varphi}=\tau_{\varphi z}=0 \\ &\varphi=\alpha/2: \quad \sigma_\varphi=-q(r,z), \quad \tau_{r\varphi}=-\mu q(r,z) \quad (r,z)\in\Omega \\ &\sigma_\varphi=\tau_{r\varphi}=0 \quad (r,z)\notin\Omega; \quad \tau_{\varphi z}=0 \end{aligned} \tag{1.1}$$

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In addition, the stresses on the wedge vanish at infinity. The solution of the Lamé equilibrium equations in r, φ, z coordinates can be expressed in terms of three arbitrary harmonic functions $\Phi_n = \Phi_n(r, \varphi, z)$ ($n = 0, 1, 2$) by the formulae [1, 2]

$$\begin{aligned}
u_r &= \frac{\partial \Phi_0}{\partial r} + \frac{1}{4(1-\nu)}\frac{\partial}{\partial r}(r\omega_1) - \omega_1, \quad \omega_1 = \sin\varphi\Phi_1 - \cos\varphi\Phi_2 \\
u_\varphi &= \frac{1}{r}\frac{\partial \Phi_0}{\partial \varphi} + \frac{1}{4(1-\nu)}\frac{\partial \omega_1}{\partial \varphi} - \omega_2, \quad \omega_2 = \cos\varphi\Phi_1 + \sin\varphi\Phi_2 \\
u_z &= \frac{\partial \Phi_0}{\partial z} + \frac{r}{4(1-\nu)}\frac{\partial \omega_1}{\partial z}
\end{aligned} \tag{1.2}$$

Using Hooke's law, from (1.2) we obtain expressions for the stresses [2, p. 147, formula (2)]. The harmonic functions Φ_n will be sought in the form of Fourier–Kontorovich–Lebedev integrals in the complex plane [2, p. 147, formula (3)]. Using a well-known technique [1, 2], we can obtain a solution of boundary-value problem (1.1) in the form (1.2), where (we change to real Kontorovich–Lebedev integrals, $n = 0, 1, 2$ and $m = 1, 2$)

$$\begin{aligned}
&\Phi_n(r,\varphi,z) = \frac{1}{\pi^3}\int_0^\infty\int_0^\infty \operatorname{sh}\pi\tau[A_n(\tau,\beta)\operatorname{ch}\varphi\tau + B_n(\tau,\beta)\operatorname{sh}\varphi\tau]K_{i\tau}(\beta r)\cos\beta z\,d\tau\,d\beta \\
&A_0(\tau,\beta) = -\frac{1-2\nu}{\beta\operatorname{sh}(\alpha\tau/2)}\int_0^\infty W_1(u)\psi_1(u,\beta)\frac{\operatorname{sh}(\pi u/2)}{\Delta_+(u,\tau)}du \\
&B_0(\tau,\beta) = \frac{1-2\nu}{\beta\operatorname{ch}(\alpha\tau/2)}\int_0^\infty W_2(u)\psi_2(u,\beta)\frac{\operatorname{sh}(\pi u/2)}{\Delta_+(u,\tau)}du \\
&A_1(\tau,\beta) = R_1(\tau)C_2(\tau,\beta) - S(\tau,\beta)\left[\frac{\tau R_1(\tau)}{1-2\nu} + S_1(\tau)\right] \\
&B_1(\tau,\beta) = R_2(\tau)C_1(\tau,\beta) - S(\tau,\beta)\left[\frac{\tau R_2(\tau)}{1-2\nu} + S_2(\tau)\right] \\
&A_2(\tau,\beta) = S_2(\tau)C_1(\tau,\beta) - S(\tau,\beta)\left[\frac{\tau S_2(\tau)}{1-2\nu} - R_2(\tau)\right] \\
&B_2(\tau,\beta) = S_1(\tau)C_2(\tau,\beta) - S(\tau,\beta)\left[\frac{\tau S_1(\tau)}{1-2\nu} - R_1(\tau)\right] \\
&C_m(\tau,\beta) = (-1)^m 2(1-\nu)\frac{W_m(\tau)}{\operatorname{ch}(\pi\tau/2)}\psi_m(\tau,\beta), \quad \psi_m(\tau,\beta) = F_m^*(\tau,\beta) + \psi_m^*(\tau,\beta) \\
&S(\tau,\beta) = -\mu\iint_\Omega \frac{q(x,y)}{G}K_{i\tau}(\beta x)\cos\beta y\,dx\,dy \\
&F_m^*(\tau,\beta) = -\iint_\Omega \frac{q(x,y)}{G}F_m(\tau,\beta x)\cos\beta y\,dx\,dy \\
&F_m(\tau,\beta x) = \left[1 - \frac{\mu f_m(\tau)}{2(1-\nu)(1-2\nu)}\right]\operatorname{ch}\frac{\pi\tau}{2}K_{i\tau}(\beta x) + \\
&+\frac{\mu}{2(1-\nu)}\operatorname{ch}\frac{\pi\tau}{2}\int_0^\infty h_m(t)K_{it}(\beta x)\frac{\operatorname{sh}\pi t}{\Delta_-(t,\tau)}dt \\
&R_{1,2}(\tau) = \begin{Bmatrix}\operatorname{ch}(\alpha\tau/2)\\ \operatorname{sh}(\alpha\tau/2)\end{Bmatrix}\frac{2\cos(\alpha/2)}{\operatorname{ch}\alpha\tau \pm \cos\alpha}, \quad S_{1,2}(\tau) = \begin{Bmatrix}\operatorname{sh}(\alpha\tau/2)\\ \operatorname{ch}(\alpha\tau/2)\end{Bmatrix}\frac{2\sin(\alpha/2)}{\operatorname{ch}\alpha\tau \pm \cos\alpha} \\
&W_{1,2}(\tau) = \pm\frac{\operatorname{ch}\alpha\tau \mp \cos\alpha}{\operatorname{sh}\alpha\tau \pm \tau\sin\alpha}, \quad h_{1,2}(\tau) = \frac{(1-2\nu)\operatorname{sh}\alpha\tau \mp \tau\sin\alpha}{\operatorname{ch}\alpha\tau \mp \cos\alpha} \\
&f_{1,2}(\tau) = \frac{\pm\tau}{W_{1,2}(\tau)} \pm \frac{2(1-\nu)(1-2\nu)\sin\alpha}{\operatorname{ch}\alpha\tau \mp \cos\alpha}, \quad \Delta_\pm(u,\tau) = \operatorname{ch}\pi u \pm \operatorname{ch}\pi\tau
\end{aligned} \tag{1.3}$$

Here $K_{i\tau}(x)$ is the MacDonald function [14]; the upper sign (function) corresponds to the first subscript while the lower sign corresponds to the second subscript. The functions $\psi_m^*(\tau,\beta)$ $(m = 1, 2)$ are found from the Fredholm integral equations of the second kind $(0 \leqslant \tau < \infty)$

$$\psi_m^*(\tau,\beta) = (1-2\nu)\int_0^\infty L_m(\tau,u)[\psi_m^*(u,\beta) + F_m^*(u,\beta)]du$$

$$L_m(\tau,u) = 2\,\mathrm{ch}\frac{\pi\tau}{2}\,\mathrm{sh}\frac{\pi u}{2}W_m(u)\int_0^\infty \frac{\mathrm{sh}\,\pi t g_m(t)}{\Delta_+(t,\tau)\Delta_+(t,u)}dt \tag{1.4}$$

$$g_{1,2}(\tau) = \left\{\begin{matrix}\mathrm{cth}(\alpha\tau/2)\\ \mathrm{th}(\alpha\tau/2)\end{matrix}\right\}\frac{\sin^2\alpha}{\mathrm{ch}\,\alpha\tau \mp \cos 2\alpha}$$

The formal solution of Eqs (1.4) can be written in the form of functional series in powers of $(1-2\nu)$, which converge uniformly in the space of continuous functions, bounded along the semiaxis [3]. When performing practical calculations, to solve Eqs (1.4) one must use the method of mechanical quadratures and the Gauss quadrature formula [12]. The singular integrals in (1.3) for the functions $F_m(\tau, \beta x)$ $(m = 1, 2)$ are calculated by standard regularization after subdividing the infinite interval of integration in order to localize the singularity in a finite interval.

Using (1.2)–(1.4) and Hooke's law one can calculate the displacement vector and the stress tensor in the three-dimensional wedge with conditions (1.1). These formulae generalize the well-known solutions of the Boussinesq and Cerruti problems for a half-space [4].

We will show this for the displacements of the boundary of the half-space by putting $\alpha = \pi$ and $\varphi = \pi/2$ in (1.2)–(1.4). When $\alpha = \pi$ we have

$$\psi_{1,2}^*(\tau,\beta) \equiv 0, \quad W_1(\tau) = \mathrm{cth}\frac{\pi\tau}{2}, \quad W_2(\tau) = -\mathrm{th}\frac{\pi\tau}{2}, \quad f_1(\tau) = \tau\,\mathrm{th}\frac{\pi\tau}{2}$$

$$f_2(\tau) = \tau\,\mathrm{cth}\frac{\pi\tau}{2}, \quad h_1(\tau) = (1-2\nu)\mathrm{th}\frac{\pi\tau}{2}, \quad h_2(\tau) = (1-2\nu)\mathrm{cth}\frac{\pi\tau}{2} \tag{1.5}$$

and for the normal displacement on the boundary of the half-space we obtain the expression

$$u_\varphi = -\iint_\Omega \frac{q(x,y)}{\theta}V(x,y,r,z)dxdy, \quad \theta = \frac{G}{1-\nu}$$

$$V(x,y,r,z) = \frac{2}{\pi^3}\int_0^\infty\int_0^\infty \mathrm{ch}\,\pi u K_{iu}(\beta x)K_{iu}(\beta r)\cos\beta y\cos\beta z d\beta du +$$

$$+\frac{\mu(1-2\nu)}{\pi^3 2(1-\nu)}\int_0^\infty\int_0^\infty \mathrm{sh}\,\pi u\left[\mathrm{cth}\frac{\pi u}{2}\int_0^\infty \mathrm{th}\frac{\pi t}{2}\mathrm{sh}\,\pi t\frac{K_{it}(\beta x)}{\Delta_-(t,u)}dt + \right. \tag{1.6}$$

$$\left. +\mathrm{th}\frac{\pi u}{2}\int_0^\infty \mathrm{cth}\frac{\pi t}{2}\mathrm{sh}\,\pi t\frac{K_{it}(\beta x)}{\Delta_-(t,u)}dt\right]K_{iu}(\beta r)\cos\beta y\cos\beta z d\beta du = \xi_1 - \frac{\mu(1-2\nu)(r-x)}{2(1-\nu)}\xi_2$$

$$\xi_m = \frac{1}{4\pi}\left(\frac{1}{R_-^m} + \frac{1}{R_+^m}\right), \quad R_\pm = [(r-x)^2 + (z \pm y)^2]^{1/2}$$

which agrees, taking the evenness of the problem with respect to z into account, with the well-known formulae [4, pp. 276 and 279 when $z = 0$]; not that in [4, Fig. 9.4] the unit vector $\mathbf{e}_z$ is directed opposite to the unit vector $\mathbf{e}_\varphi$ introduced above and was chosen to be in the opposite direction to the shear load). When calculating the quadratures in (1.6) one must use formulae 2.5.30.8 [13], 2.16.14.4 (with $\nu = 0$, omitting the quantity c in the argument of the MacDonald function), 2.16.48.20 and 2.16.52.6 [14], and also the value of the integral $(\beta > 0)$

$$\int_0^\infty \mathrm{th}\,u\cos(\beta r\,\mathrm{sh}\,u)\sin(\beta x\,\mathrm{sh}\,u)du = \frac{\pi}{4}(\mathrm{sgn}(x-r)e^{-\beta|x-r|} + e^{-\beta|x+r|}) \tag{1.7}$$

and the fact that, for the odd function $W(u)$

$$
\begin{aligned}
&2\int_0^\infty \frac{W(u)\operatorname{sh}\pi u}{\Delta_-(u,t)}du = \frac{1}{\pi\operatorname{ch}(\pi t/2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{ch}\frac{\pi u}{2}W(u)\operatorname{th}x\sin xu\cos xt\,dx\,du = \\
&= -\frac{1}{\pi\operatorname{sh}(\pi t/2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\operatorname{sh}\frac{\pi u}{2}W(u)\operatorname{th}x\cos xu\sin xt\,dx\,du
\end{aligned}
\tag{1.8}
$$

The representations of the singular integral (1.8) are proved using a Fourier integral transformation.
Similarly, taking into account the value of the integral

$$
\int_0^\infty \frac{u\operatorname{sh}\pi u K_{iu}(x)}{\Delta_+(u,t)}du = xK_{it}(x)
\tag{1.9}
$$

we find that when $\alpha = \pi$ and $\varphi = \pi/2$

$$
\begin{aligned}
&\Phi_0\left(r,\frac{\pi}{2},z\right) = -\iint_\Omega \frac{q(x,y)}{\theta}\Phi(x,y,r,z)\,dx\,dy \\
&\Phi(x,y,r,z) = -\frac{1-2\nu}{\pi^3(1-\nu)}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\operatorname{ch}\pi t\operatorname{ch}\pi u+1}{\Delta_+(t,u)}K_{it}(\beta r)K_{iu}(\beta x)\frac{\cos\beta z\cos\beta y}{\beta}d\beta\,dt\,du + \\
&+\frac{\mu[x+(1-2\nu)^2(r-x)]}{4(1-\nu)^2}\xi_1 \\
&\omega_1 = \sin\frac{\pi}{2}\Phi_1\left(r,\frac{\pi}{2},z\right) = \frac{\mu}{1-\nu}\iint_\Omega \frac{q(x,y)}{\theta}\xi_1\,dx\,dy
\end{aligned}
\tag{1.10}
$$

The integral with respect to the variable β diverges, but here, according to expression (1.2), we require the partial derivatives of Φ_0 with respect to r and z, which are calculated using representations 9.6.22 [12], the relation

$$
\operatorname{ch}\pi t\operatorname{ch}\pi u + 1 = 4\operatorname{ch}^2\frac{\pi t}{2}\operatorname{ch}^2\frac{\pi u}{2} - (\operatorname{ch}\pi t + \operatorname{ch}\pi u)
\tag{1.11}
$$

and the values of the integrals 2.5.6.4, 2.5.48.2 [13] and 2.16.48.1 [14]. As a result, we have on the boundary of the half-space

$$
\begin{aligned}
&u_r = -\iint_\Omega \frac{q(x,y)}{\theta}U(x,y,r,z)\,dx\,dy, \quad u_z = -\iint_\Omega \frac{q(x,y)}{\theta}W(x,y,r,z)\,dx\,dy \\
&U(x,y,r,z) = \frac{(1-2\nu)(r-x)}{2(1-\nu)}\xi_2 + \mu\left[\xi_1 + \frac{\nu(r-x)^2}{1-\nu}\xi_3\right] \\
&W(x,y,r,z) = \frac{(1-2\nu)(z-y)}{2(1-\nu)}\xi_2 + \frac{\mu\nu(r-x)(z-y)}{1-\nu}\xi_3
\end{aligned}
\tag{1.12}
$$

which is identical with well-known formulae ([4, pp.276 and 279 with $z = 0$]; note that in [4] (Fig. 9.4) the unit vectors $\mathbf{e}_x$ and $\mathbf{e}_y$ coincide with the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_z$ introduced above, respectively).

2. THE CONTACT PROBLEM FOR A WEDGE

We will investigate the quasi-static contact problem when a rigid punch, initially imbedded into the face of an elastic wedge, begins to move fairly slowly along this face (without inclination) in a direction perpendicular to the edge of the wedge. The punch is an elliptic paraboloid highly prolate along the edge, and hence it can be assumed approximately that the friction forces are collinear with the direction of motion (and directed opposite to it). The formulation of the problem extends the well-known case of the motion of a punch over a half-space [5] to the case of a wedge of aperture angle α, one face of which is stress-free. The problem is symmetrical about the z coordinate.

We will use the solution of boundary-value problem (1.1) obtained above. By satisfying the boundary condition of contact between the bodies $u_\varphi(r, \alpha/2, z) = -[\delta - f(r, z)]$, $(r, z) \in \Omega$, where δ is the settlement of the punch and $f(r, z) = (r - a)^2/(2R_1) + z^2/(2R_2)$ is the shape of the punch base ($R_1 \ll R_2$), and of the relatively unknown normal contact pressure $\sigma_\varphi(r, \alpha/2, z) = -q(r, z)$ in the unknown contact area $(r, z) \in \Omega$, we obtain the following integral equation (see also formula (4) in [3])

$$\iint\limits_{\Omega} q(x,y)\left[\frac{1}{R_-} - \mu\frac{1-2\nu}{2(1-\nu)}\frac{r-x}{R_-^2} + K(x,y,r,z)\right]dxdy = 2\pi\theta[\delta - f(r,z)]$$

$$K(x,y,r,z) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{sh}\frac{\pi u}{2} W(u,\beta x)K_{iu}(\beta r)\cos\beta(y-z)d\beta du$$

$$W(u,\beta x) = W_1(u)\psi_1^*(u,\beta x) - W_2(u)\psi_2^*(u,\beta x) + 2\operatorname{ch}\frac{\pi u}{2}K_{iu}(\beta x)[W_0(u) -$$

$$-\operatorname{cth}\pi u + \mu f_0(u)] + \frac{\mu}{2(1-\nu)}\operatorname{ch}\frac{\pi u}{2}\int_0^\infty \{W_1(u)h_1(t) - W_2(u)h_2(t) - \qquad (2.1)$$

$$-(1-2\nu)\left[\operatorname{cth}\frac{\pi u}{2}\operatorname{th}\frac{\pi t}{2} + \operatorname{th}\frac{\pi u}{2}\operatorname{cth}\frac{\pi t}{2}\right]\right\}\operatorname{sh}\pi t\frac{K_{it}(\beta x)}{\Delta_-(t,u)}dt$$

$$W_0(u) = \frac{\operatorname{sh}2\alpha u + u\sin 2\alpha}{\operatorname{ch}2\alpha u - 2u^2\sin^2\alpha - 1}, \quad f_0(u) = \frac{2u\sin^2\alpha}{\operatorname{ch}2\alpha u - 2u^2\sin^2\alpha - 1}$$

The functions $\psi_m^*(u, \beta x)$ ($m = 1, 2$) satisfy the Fredholm integral equations of the second kind (1.4), in which, instead of the functions $F_m^*(u, \beta)$ we must substitute the functions $F_m(u, \beta x)$ (1.3). In the kernel of integral equation (2.1), using formulae (1.6), we can explicitly separate out the singular part to improve the convergence of the integrals. When the Coulomb friction coefficient $\mu > 0$, the punch begins to move towards the edge of the wedge, and when $\mu < 0$ it begins to move away from the edge. When $\alpha = \pi$ Eq. (2.1) is identical with Eq.(3) [5] (taking into account the fact that the motion is in the negative direction of the r axis when $\mu > 0$).

To solve integral equation (2.1) with the condition $q(r, z) = 0$, $(r, z) \in \partial\Omega$, we will use the method of non-linear boundary integral equations [6, 7], which enables us to determine the normal contact pressures and the contact area simultaneously. The main properties of the integral operator, generated by the kernel of Eq.(2.1) when $\mu = 0$ [7] (strict positiveness and complete continuity) are also preserved when $\mu \neq 0$ (this was pointed out previously in [6, p.19] for the case $\alpha = \pi$). Hence, the well-known results concerning the existence and uniqueness and a method of constructing the solution of Eq.(2.1) when $\mu = 0$ [7], can be transferred completely to the case of calculating the friction forces being considered here.

We will further use the dimensionless notation (2.1) [7] (we omit the primes, see also the paragraph following formulae (2.1) [7]). The calculation were carried out for angles $\alpha = 70°, 90°, 110°$ and $180°$.

An analysis of the results shows that when $\alpha = 180°$ the value of $P(\delta)$ [7] is practically independent of μ. This can be explained as follows. When $\alpha = 180°$ the solution of Eq.(2.1) can be sought in the form of a series in powers of the small parameter $\varepsilon_* = \mu(1 - 2\nu)/(2 - 2\nu)$ [5]. Dropping terms of the order of ε_*^2 (when $\mu = 0.2$ and $\nu = 0.3$ we have $\varepsilon_*^2 \approx 0.003$), we obtain $q(r, z) = q_0(r, z) \mp \varepsilon_* q_1(r, z) + O(\varepsilon_*^2)$, where the minus (plus) sign is taken for the case when the punch moves in the positive (negative) direction of the r axis. The function $q_0(r, z)$ satisfies the integral equation of the contact problem for a half-space without friction, and the function $q_1(r, z)$ is expressed in terms of this function (irrespective of ε_*). The normal force P (the integral of the function $q(r, z)$ over the area Ω), is obviously independent of the direction of motion of the punch in the half-space. Hence it is clear that the integral of the function $q_1(r, z)$ over the contact area must be equal to zero. Hence, the order of the terms which describe the effect of the friction forces in the relation $P(\delta)$ will be $o(\varepsilon_*)$.

Friction will have a considerable effect on the eccentricity of the normal force (and on the moment), which ensures motion of the punch without inclination. We know that, in the case of the axisymmetrical contact problem with friction [15], the friction forces generally have no effect on the relation $P(\delta)$ (two independent strain and stress systems were obtained: from one of these the normal force, which is independent of μ, is obtained, and from the other the moment, which depends on the friction). The smaller the angle of the wedge the more appreciable is the effect of friction and the direction of motion on the relation $P(\delta)$.

In Table 1 (the quadrant $\alpha = 90°$) we show values of the normal force $P \times 10^3$ as a function of the settlement $\delta \times 10^3$ for different friction coefficients μ. Here, in the notation employed earlier [7]

Table 1

μ	$\delta \times 10^3 = 4$	4.5	5	5.5	6	6.5
-0.2	0.531	0.632	0.740	0.849	0.965	1.09
-0.1	0.510	0.605	0.708	0.812	0.920	1.03
0	0.491	0.581	0.678	0.777	0.879	0.986
0.1	0.474	0.559	0.650	0.746	0.841	0.942
0.2	0.457	0.539	0.625	0.716	0.808	0.903

$A = 0.1, B = 0.005, \varepsilon = 0.15, \lambda = 1$ and $\nu = 0.3$ ($\gamma = 0$).

The closer the punch is to the edge of the wedge with aperture angle $\alpha < \pi$, i.e. the smaller the value of λ (7), the smaller the value of $P(\delta)$, which is due to the increase in the compliance of the elastic material. This conclusion, which is well known when there is no friction [7], still holds for fixed $\mu \neq 0$. If the punch moves towards the edge of the wedge ($\mu > 0$; the value of λ is fixed), the value of $P(\delta)$ is less (the punch is easier to press in), than when $\mu = 0$ and with the same value of λ. If the punch recedes from the edge ($\mu < 0$; the value of λ is fixed), the value of $P(\delta)$ is greater than when $\mu = 0$ and with the same value of λ. This increase in the force when $\mu < 0$ occurs due to the increase in the maximum normal contact pressure; the contact area when $\mu < 0$ may be less than when $\mu = 0$, which in turn is less than the area of the region Ω when $\mu > 0$ (for example, when $\alpha = 70°$, $\lambda \approx 0$, $\delta \times 10^3 = 6.5$ and the values of the remaining parameters as in the table). When $\delta = \text{const}$ the motion of the punch in the direction of the edge of the wedge ($\mu > 0$) prevents breaking down the contact in the neighbourhood of the edge (the edge moves away from the punch), observed when $\mu = 0$ and $\lambda \approx 0$ for fairly acute angles α [7].

3. ANALYSIS OF THE EFFECTIVE STRESS

After solving the contact problem with a specified function $q(r, z)$ and contact area Ω it is possible to determine the important role played in applications [10,11] of the dimensionless effective stress $\sigma_e' = \sigma_e/(2\pi\theta)$. Using the optical analogy method it was established in [15, p. 67] that for friction under a smooth punch the zone of maximum shear stresses rises close to the boundary of the elastic half-plane. As an example (within the framework of the idea of surface strength) we will calculate σ_e' on the axis of symmetry of the contact area using the first formula of (3.1) [7] and the following formulae (we omit the prime, compare with (3.1)–(3.3) in [7])

$$\sigma_e = 2^{-1/2}[(\sigma_r - \sigma_\varphi)^2 + (\sigma_\varphi - \sigma_z)^2 + (\sigma_z - \sigma_r)^2 + 6\tau_{r\varphi}^2]^{1/2} \tag{3.1}$$

$$\sigma_r = \frac{\nu}{1-\nu}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) + \frac{\partial u_r}{\partial r} - \frac{\nu}{1-\nu}q(r,0), \quad \sigma_\varphi = -q(r,0)$$

$$\sigma_z = \frac{\nu}{1-\nu}\left(\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}\right) + \frac{\partial u_z}{\partial z} - \frac{\nu}{1-\nu}q(r,0), \quad \tau_{r\varphi} = -\mu q(r,0)$$

$$\frac{\partial u_r}{\partial r} = -\frac{1-2\nu}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty E_1(\beta,t,u)K_*(\beta,t,r_0)\,d\beta\,dt\,du -$$

$$-\frac{1}{\pi^3}\int_0^\infty\int_0^\infty E_2(\beta,t)\left[\frac{r_0}{2}K_*(\beta,t,r_0) + (1-2\nu)\operatorname{Re}K_{1+it}(\beta r_0)\right]d\beta\,dt -$$

$$-Q(r_0) - (1-2\nu)q(r,0) - \frac{\mu(1-\nu)}{\pi}\iint_\Omega \frac{1}{[(r_0-x)^2+y^2]^{1/2}}\frac{\partial}{\partial x}q(x,y)\,dx\,dy$$

$$\frac{\partial u_z}{\partial z} = \frac{1-2\nu}{\pi^3}\int_0^\infty\int_0^\infty\int_0^\infty E_1(\beta,t,u)\beta K_{it}(\beta r_0)\,d\beta\,dt\,du +$$

$$+\frac{r_0}{2\pi^3}\int_0^\infty\int_0^\infty E_2(\beta,t)\beta K_{it}(\beta r_0)\,d\beta\,dt + Q(r_0)$$

$$Q(r_0) = \frac{1-2\nu}{2\pi}\iint_\Omega \frac{y}{(r_0-x)^2+y^2}\frac{\partial}{\partial y}q(x,y)\,dx\,dy +$$

$$+\frac{\mu\nu}{\pi}\iint_\Omega \frac{(r_0-x)y}{[(r_0-x)^2+y^2]^{3/2}}\frac{\partial}{\partial y}q(x,y)\,dx\,dy, \quad r_0 = \begin{cases} r+\lambda\ (\lambda > \varepsilon) \\ r+\varepsilon\ (\lambda \leq \varepsilon)\end{cases}$$

$$K_*(\beta,t,r_0)=\beta K_{it}(\beta r_0)-[t\operatorname{Im}K_{1+it}(\beta r_0)-\operatorname{Re}K_{1+it}(\beta r_0)]/r_0$$

$$E_1(\beta,t,u)=\frac{\operatorname{sh}\pi t\operatorname{sh}\pi u}{\Delta_+(t,u)}\left\{W_1(u)\operatorname{cth}\frac{\alpha t}{2}\frac{\Psi_1(u,\beta)}{\operatorname{ch}(\pi u/2)}-W_2(u)\operatorname{th}\frac{\alpha t}{2}\frac{\Psi_2(u,\beta)}{\operatorname{ch}(\pi u/2)}-\right.$$

$$-\operatorname{cth}\frac{\pi u}{2}\operatorname{cth}\frac{\pi t}{2}\left[\left(1-\frac{\mu u\operatorname{th}(\pi u/2)}{2(1-\nu)(1-2\nu)}\right)S_*(u,\beta)+\frac{\mu(1-2\nu)}{2(1-\nu)}\times\right.$$

$$\left.\times\int_0^\infty\operatorname{th}\frac{\pi\tau}{2}\operatorname{sh}\pi\tau\frac{S_*(\tau,\beta)}{\Delta_-(\tau,u)}d\tau\right]-\operatorname{th}\frac{\pi u}{2}\operatorname{th}\frac{\pi t}{2}\left[\left(1-\frac{\mu u\operatorname{cth}(\pi u/2)}{2(1-\nu)(1-2\nu)}\right)S_*(u,\beta)+\right.$$

$$\left.\left.+\frac{\mu(1-2\nu)}{2(1-\nu)}\int_0^\infty\operatorname{cth}\frac{\pi\tau}{2}\operatorname{sh}\pi\tau\frac{S_*(\tau,\beta)}{\Delta_-(\tau,u)}d\tau\right]\right\}$$

$$E_2(\beta,t)=-2\beta\operatorname{sh}\pi t\left\{\frac{\sin\alpha}{\operatorname{ch}(\pi t/2)}\left[\frac{\Psi_1(t,\beta)}{\operatorname{sh}\alpha t+t\sin\alpha}-\frac{\Psi_2(t,\beta)}{\operatorname{sh}\alpha t-t\sin\alpha}\right]-\right.$$

$$\left.-\frac{\mu}{1-\nu}S_*(t,\beta)\left[\frac{\operatorname{sh}2\alpha t-t(1-2\nu)^{-1}\sin 2\alpha}{\operatorname{ch}2\alpha t-\cos 2\alpha}-\operatorname{cth}\pi t\right]\right\}$$

$$S_*(t,\beta)=-\iint_\Omega q(r,z)K_{it}(\beta r_0)\cos\beta z\,dr\,dz$$

Here all the components of the stress tensor relate to $2\pi\theta$, and the functions $\Psi_m(t,\beta)$ ($m=1,2$) are found from (1.3), where we must put $G=1$. To improve the convergence of the integrals in (3.1) for $\partial u_r/\partial r$ and $\partial u_z/\partial z$ we must explicitly separate out the terms corresponding to the case $\alpha=\pi$, where we use the first formula of (3.6) [7], the values of the integrals indicated above and integration by parts.

For $\alpha=\pi$ we have $E_1(\beta,t,u)\equiv E_2(\beta,t)\equiv 0$ and, integrating by parts, we can show that the stresses σ_r and σ_z (3.1) on the boundary of the half-space are identical with the well-known formulae [4, pp. 277 and 279, for $z=0$].

If we put $\mu=0$ and $\alpha=\pi$ in (3.1) and assume that the function $q(r,z)$ is defined in the elliptic region Ω by (3.4) [7], then at the initial contact point (the centre of the ellipse) we obtain the well-known formula (3.5) [7] for σ_e. For a sufficiently prolate ellipse (this case is also considered here) this value of σ_e at the centre of the ellipse will be a maximum on the contact surface and will considerably exceed the value of σ_e at the edge of the ellipse [16, p. 77]. If the shape of the ellipse Ω is close to a circle, the maximum value of σ_e on the contact surface occurs at the edge of the semi-major axis, but it will only slightly exceed the value of σ_e at the initial contact point [16, p. 77].

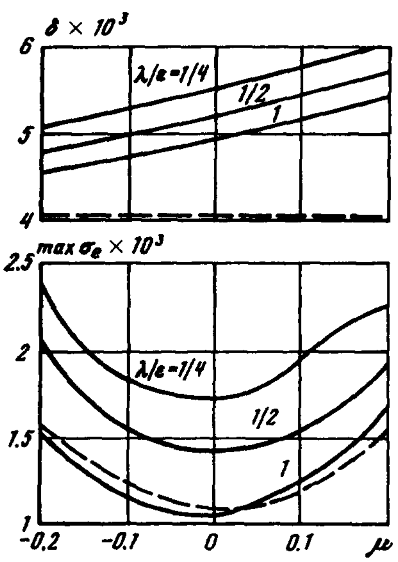

Fig. 1.

Figure 1 shows graphs of the settlement $\delta \times 10^3$ and the maximum effective stress on the axis of symmetry of the contact area $\sigma_e \times 10^3$ for a constant indenting force $P \times 10^3 = 0.583$ as a function of the friction coefficient μ (and the direction of motion of the punch) for a half-space (the dashed curves) and for a wedge with an aperture angle $\alpha = 110°$ (the continuous curves) for various values of λ, which characterizes the degree of closeness of the punch to the edge of the wedge [7]. The values of the elasticity parameters are taken in the table. For a half-space (and for a wedge with values of λ and α that are not too small), when the friction forces are taken into account, max σ_e as before is reached at the initial contact point, increasing as $|\mu|$ increases. For a constant indenting force close to the edge of the wedge (for sufficiently small λ) the point where max σ_e is reached begins to shift from the initial contact point, as a rule, towards the edge of the wedge when $\mu < 0$ (the friction forces are directed towards the edge) and the other way when $\mu > 0$ (the friction forces are directed away from the edge). When $\alpha = 70°$ and $\lambda = \varepsilon$ and values of the other parameters as in the calculations for the figure, contact breaks down in the neighbourhood of the edge (due to the considerable increase in the settlement for a constant force).

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